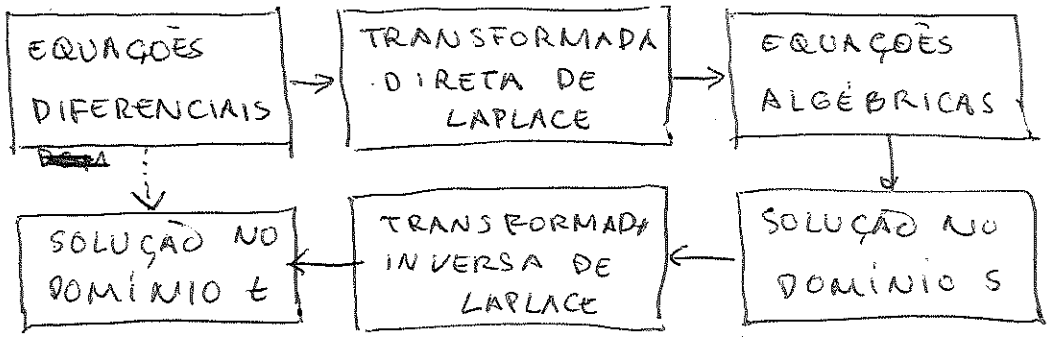
4 aulas

Capítulo 5. Transformada de Laplace



5.1. Definições

a) transformadas de Laplace

A transformada direta de Laplace é a operação

$$X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt = \mathcal{L}\left[x(t)\right]$$

e a ~~opera~~ transformada inversa, é a operação

$$x(t) = \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds = \mathcal{L}^{-1}\left[X(s)\right]$$

b) sinal no domínio do tempo contínuo t

$x(t)$ é uma função real da variável real t (e contínua) que representa o sinal no domínio do tempo.

c) sinal no domínio da frequência complexa s

$X(s)$ é uma função real da variável complexa $s = \sigma + j2\pi f$ que representa o sinal no domínio da frequência complexa.

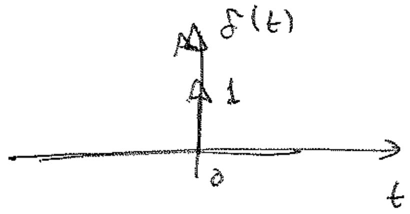
d) par transformado $x(t) \leftrightarrow X(s)$

A correspondência entre $x(t)$ e $X(s)$ é unívoca portanto ambas representam o mesmo sinal em domínios distintos.

Exercício 5.1.

Calcule as transformadas diretas de Laplace para os sinais abaixo.

a) impulso $x(t) = \delta(t)$

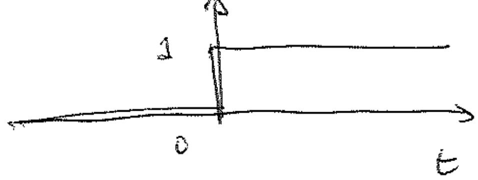


$$X(s) = \int_0^\infty \delta(t)\, e^{-st}\, dt$$

$t=0$

$$X(s) = \int_0^\infty \delta(t)\, dt = 1$$

b) degrau $x(t) = u(t)$



$$X(s) = \int_0^\infty \underbrace{u(t)}_{1}\, e^{-st}\, dt$$

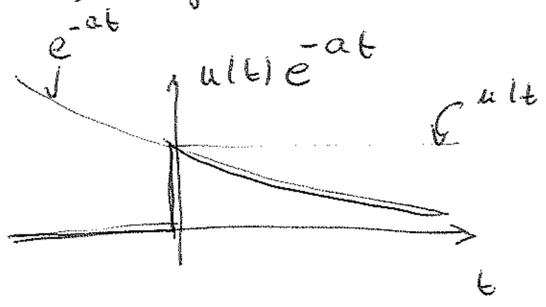
$$X(s) = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$\lim_{t\to\infty} e^{-st} = \lim_{t\to\infty} e^{-(\sigma + j2\pi f)t} = \lim_{t\to\infty} e^{-\sigma t} \cdot e^{-j2\pi f t}$$

$$= \lim_{t\to\infty} e^{-\sigma t} \cdot \left[\cos(2\pi f t) - j\,\text{sen}(2\pi f t)\right] = 0 \text{ se } \sigma > 0$$

$$X(s) = \frac{0-1}{-s} = \frac{1}{s} \quad \text{com } \operatorname{Re}[s] > 0$$

c) exponencial decrescente $\quad x(t) = u(t)\, e^{-at}, \quad a > 0$



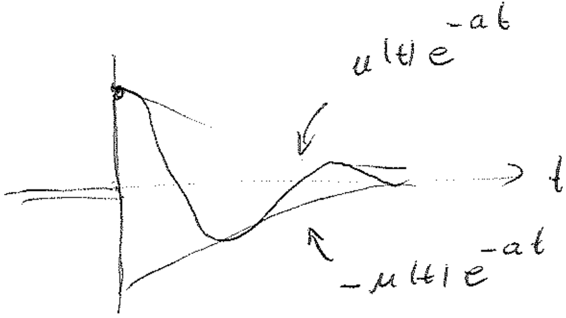
$$X(s) = \int_0^{\infty} u(t)\, e^{-at} e^{-st}\, dt$$

$$X(s) = \int_0^{\infty} e^{-(s+a)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$X(s) = \frac{0-1}{-(s+a)} = \frac{1}{s+a}$$

d) cossenoide amortecida $\quad x(t) = u(t)\, e^{-at} \cos(bt)$

$a > 0, \ b > 0.$



$$X(s) = \int_0^{\infty} u(t)\, e^{-at} \cos(bt)\, e^{-st}\, dt$$

$$X(s) = \int_0^{\infty} e^{-at}\, \frac{e^{jbt} + e^{-jbt}}{2}\, e^{-st}\, dt$$

$$X(s) = \frac{1}{2} \int_0^{\infty} e^{-(s+a-jb)t}\, dt + \frac{1}{2} \int_0^{\infty} e^{-(s+a+jb)t}\, dt$$

$$X(s) = \frac{1}{2} \left[\frac{e^{-(s+a-jb)t}}{-(s+a-jb)} \right]_0^{\infty} + \frac{1}{2} \left[\frac{e^{-(s+a+jb)t}}{-(s+a+jb)} \right]_0^{\infty}$$

$$X(s) = \frac{1}{2}\,\frac{0-1}{-(s+a-jb)} + \frac{1}{2}\,\frac{0-1}{-(s+a+jb)}$$

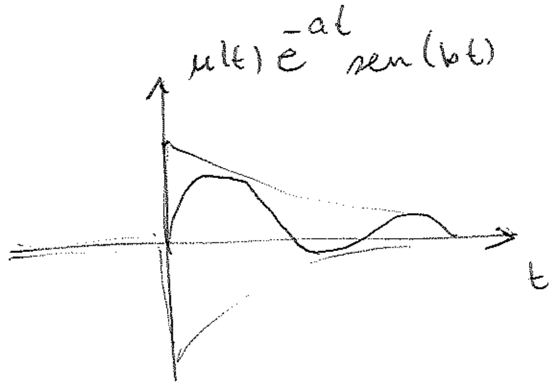
$$X(s) = \frac{1/2}{s+a-jb} + \frac{1/2}{s+a+jb}$$

$$X(s) = \frac{\frac{1}{2}(s+a) + j\frac{b}{2} + \frac{1}{2}(s+a) - j\frac{b}{2}}{(s+a)^2 - (jb)^2}$$

$$X(s) = \frac{s+a}{(s+a)^2 + b^2}$$

e) senoide amortecida $\quad x(t) = u(t)\,e^{-at}\,\text{sen}(bt)$.

$a > 0, \; b > 0$.



$$X(s) = \frac{b}{(s+a)^2 + b^2}$$

5.2. Propriedades da transformada de Laplace

a) linearidade

$$y(t) = k_1 x_1(t) + k_2 x_2(t) \Leftrightarrow Y(s) = k_1 X_1(s) + k_2 X_2(s)$$

b) deslocamento

$$y(t) = x(t-k) \Leftrightarrow Y(s) = X(s)\, e^{-sk}$$

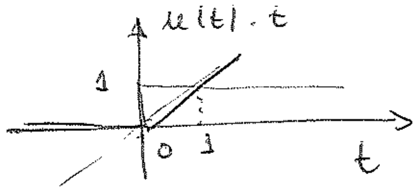
c) escalonamento

$$y(t) = x(t/k) \Leftrightarrow Y(s) = k\, X(sk) \quad , \quad k > 0$$

d) produto por t

$$y(t) = x(t) \cdot t \Leftrightarrow Y(s) = -\frac{d}{ds} X(s)$$

exemplo

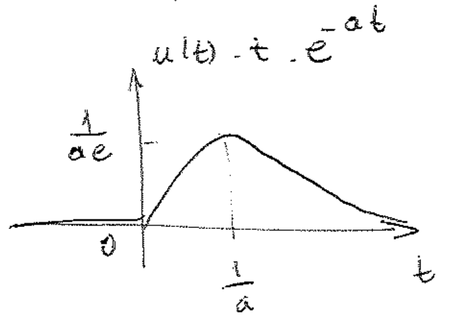


$$\Leftrightarrow -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$$

e) produto por exponencial

$$y(t) = x(t) \cdot e^{-kt} \Leftrightarrow Y(s) = X(s+k)$$

exemplo



$$\Leftrightarrow \frac{1}{(s+a)^2}$$

f) diferenciação

$$y(t) = \frac{d}{dt} x(t) \Leftrightarrow Y(s) = s\,X(s) - \underset{\substack{\uparrow \\ \text{condição inicial}}}{x(0)}$$

g) integração

$$y(t) = \int_{-\infty}^{t} x(t)\,dt \Leftrightarrow Y(s) = \frac{1}{s} X(s) + \frac{1}{s} \int_{-\infty}^{0} x(t)\,dt$$

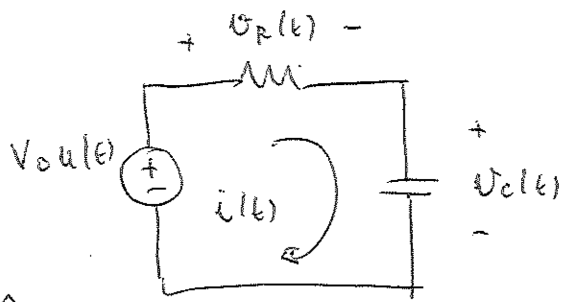
h) convolução

$$y(t) = x_1(t) * x_2(t) \Leftrightarrow Y(s) = X_1(s) \cdot X_2(s)$$

Exercício 5.2

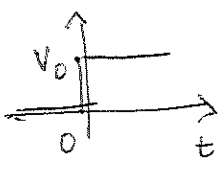
Encontre as transformadas de Laplace $I(s)$ das correntes elétricas $i(t)$ indicadas nos circuitos elétricos abaixo.

a) circuito RC



$$v_R(t) = R\,i(t)$$

$$v_c(t) = \frac{1}{C} \int_{-\infty}^{t} i(t)\,dt$$



malha no domínio t

$$V_0\, u(t) - v_R(t) - v_C(t) = 0$$

$$V_0\, u(t) - R\, i(t) - \frac{1}{C}\int_{-\infty}^{t} i(t)\, dt = 0$$

transformada de Laplace

$$V_0\, u(t) \Leftrightarrow V_0 \cdot \frac{1}{s}$$

$$R\, i(t) \Leftrightarrow R \cdot I(s)$$

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\, dt \Leftrightarrow \frac{1}{C}\left[\frac{1}{s} I(s) + \frac{1}{s}\int_{-\infty}^{0} i(t)\, dt\right]$$

$$v_c(0) = \frac{1}{C}\int_{-\infty}^{0} i(t)\, dt$$

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\, dt \Leftrightarrow \frac{I(s)}{sC} + \frac{v_c(0)}{s}$$

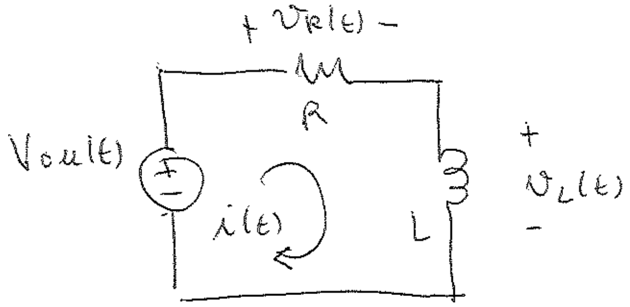
malha no domínio s

$$\frac{V_0}{s} - R\, I(s) - \frac{I(s)}{sC} - \frac{v_c(0)}{s} = 0$$

$$I(s) = \frac{\frac{V_0}{s} - \frac{v_c(0)}{s}}{R + \frac{1}{sC}} \times \frac{s}{s} = \frac{V_0 - v_c(0)}{sR + 1/C} \times \frac{1/R}{1/R}$$

$$I(s) = \frac{\frac{V_0 - v_c(0)}{R}}{s + 1/RC}$$

b) circuito RL



$$v_R(t) = R\, i(t)$$

$$v_L(t) = L \frac{d}{dt} i(t)$$

malha no domínio t

$$V_0 u(t) - v_R(t) - v_L(t) = 0$$

$$V_0 u(t) - R\, i(t) - L \frac{d}{dt} i(t) = 0$$

transformada de Laplace

$$V_0 u(t) \leftrightarrow \frac{V_0}{s}$$

$$R\, i(t) \leftrightarrow R\, I(s)$$

$$L \frac{d}{dt} i(t) \leftrightarrow L \left[s I(s) - i(0) \right]$$

malha no domínio s

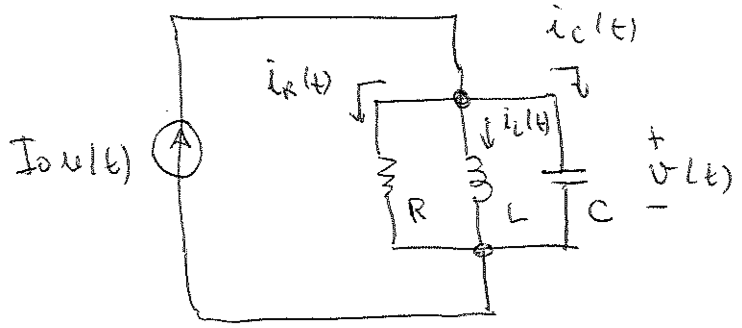
$$\frac{V_0}{s} - R I(s) - s L I(s) + L i(0) = 0$$

$$I(s) = \frac{\frac{V_0}{s} + L i(0)}{R + sL} \times \frac{s}{s} = \frac{sL i(0) + V_0}{s^2 L + sR} \times \frac{1/L}{1/L}$$

$$I(s) = \frac{s\, i(0) + V_0/L}{s^2 + s R/L}$$

Exercício 5.3

Encontre a transformada de Laplace $V(s)$ da tensão elétrica $v(t)$ indicada no circuito elétrico abaixo.



$$i_R(t) = \frac{v(t)}{R}, \quad i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt, \quad i_C(t) = C\frac{d}{dt}v(t)$$

$$V(s) = \frac{s\,v(0) + \dfrac{I_0 - i_L(0)}{C}}{s^2 + s/RC + 1/LC}$$

5.3 Função Racional

a) definição

Uma função racional em s é aquela que pode ser escrita como uma razão entre dois polinômios em s conforme

$$X(s) = \frac{b_p s^p + b_{p-1} s^{p-1} + \dots + b_1 s + b_0}{a_p s^p + a_{p-1} s^{p-1} + \dots + a_1 s + a_0}$$

b) polos

Polos são os valores de s que anulam o denominador da função racional, ou seja, as p raízes $\lambda_1, \lambda_2, \dots, \lambda_p$ da equação

$$a_p s^p + a_{p-1} s^{p-1} + \dots + a_1 s + a_0 = 0$$

c) forma fatorada

A função racional também pode ser escrita com os polos representados explicitamente na forma

$$X(s) = \frac{b_p s^p + b_{p-1} s^{p-1} + \dots + b_1 s + b_0}{a_p (s-\lambda_1)(s-\lambda_2)\dots(s-\lambda_p)}$$

d) decomposição em frações parciais

Se $X(s)$ é uma função racional do tipo

$$X(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

onde $a_3 = 1$ e o numerador tem grau menor que o denominador, então $X(s)$ pode ser decomposta em três frações parciais.

- polos distintos e reais: $\lambda_1, \lambda_2, \lambda_3$

$$X(s) = \frac{b_2 s^2 + b_1 s + b_0}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = \frac{c_1}{s-\lambda_1} + \frac{c_2}{s-\lambda_2} + \frac{c_3}{s-\lambda_3}$$

$c_1 = \left[(s-\lambda_1) X(s)\right]_{s=\lambda_1}$ $\Leftrightarrow$ resíduo do polo λ_1

$\vdots$

$c_3 = \left[(s-\lambda_3) X(s)\right]_{s=\lambda_3}$

$$x(t) = u(t)\left[c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t}\right]$$

$$\begin{cases} \lambda_p < 0 \rightarrow \text{exponencial decrescente} \\ \lambda_p = 0 \rightarrow \text{degrau} \end{cases}$$

Resposta super-amortecida.

- polos com duplicidade: $\lambda_1 = \lambda_2$, λ_3

$$X(s) = \frac{b_2 s^2 + b_1 s + b_0}{(s-\lambda_1)^2 (s-\lambda_3)} = \frac{c_1}{(s-\lambda_1)^2} + \frac{c_2}{(s-\lambda_1)} + \frac{c_3}{s-\lambda_3}.$$

$$c_1 = \left[(s-\lambda_1)^2 X(s) \right]_{s=\lambda_1}$$

$$c_2 = \left[\frac{d}{ds} (s-\lambda_1)^2 X(s) \right]_{s=\lambda_1}$$

$$c_3 = \left[(s-\lambda_3) X(s) \right]_{s=\lambda_3}$$

$$x(t) = u(t) \left[c_1 t e^{\lambda_1 t} + c_2 e^{\lambda_1 t} + c_3 e^{\lambda_3 t} \right]$$

Resposta criticamente amortecida.

- polos complexos conjugados: $\lambda_1 = \lambda_2^*$, λ_3

$$X(s) = \frac{b_2 s^2 + b_1 s + b_0}{(s-\lambda_1)(s-\lambda_1^*)(s-\lambda_3)} = \frac{c_1}{s-\lambda_1} + \frac{c_1^*}{s-\lambda_1^*} + \frac{c_3}{s-\lambda_3}$$

$c_1 = c_r + j c_i$, $\lambda_1 = -a + jb$, $a \geq 0$ e $b > 0$

$$X(s) = \frac{c_r + j c_i}{s+a-jb} + \frac{c_r - j c_i}{s+a+jb} + \frac{c_3}{s-\lambda_3}$$

$$X(s) = \frac{2c_r(s+a) - 2c_i b}{(s+a)^2 + b^2} + \frac{c_3}{s-\lambda_3}$$

$$x(t) = u(t) \left[2c_r e^{-at} \cos(bt) - 2c_i e^{-at} \operatorname{sen}(bt) + c_3 e^{\lambda_3 t} \right]$$

Resposta subamortecida (a > 0) ou não amortecida (a = 0).

Exercício 5.4

Calcule a transformada inversa de Laplace $v(t)$ da função $V(s)$ do exercício 5.3 com:

a) $R = 1/3$, $L = 1/2$, $C = 1$, $v(0) = 2$, $i_L(0) = 0$, $I_0 = 1$

$$V(s) = \frac{2s+1}{s^2+3s+2}$$

polos: $s^2+3s+2=0 \Rightarrow s = \dfrac{-3 \pm \sqrt{9-8}}{2} = \begin{cases} -1 = \lambda_1 \\ -2 = \lambda_2 \end{cases}$

fatorada: $V(s) = \dfrac{2s+1}{(s-\lambda_1)(s-\lambda_2)} = \dfrac{2s+1}{(s+1)(s+2)}$

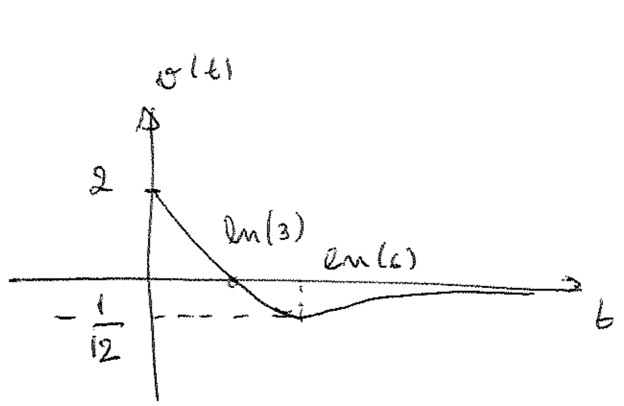
resíduos: $c_1 = \left[(s-\lambda_1)V(s)\right]_{s=\lambda_1} = \left[(s+1)\dfrac{2s+1}{(s+1)(s+2)}\right]_{s=-1}$

$$c_1 = -1$$

$$c_2 = \left[(s-\lambda_2)V(s)\right]_{s=\lambda_2} = \left[(s+2)\frac{2s+1}{(s+1)(s+2)}\right]_{s=-2}$$

$$c_2 = 3$$

$$V(s) = \frac{-1}{s+1} + \frac{3}{s+2} \Leftrightarrow v(t) = u[t]\left[-e^{-t} + 3e^{-2t}\right]$$



c) $R=1, \quad L=2/5, \quad C=1, \quad v(0)=0, \quad i_L(0)=0, \quad I_0=1$

$$V(s) = \frac{1}{s^2 + s + 5/2}$$

polos : $s^2+s+5/2=0 \Rightarrow s = \dfrac{-1 \pm \sqrt{1-10}}{2} = \begin{cases} -\frac{1}{2} + j\frac{3}{2} = \lambda_1 \\ -\frac{1}{2} - j\frac{3}{2} = \lambda_2 \end{cases}$

$\lambda_1 = \lambda_2^* = -a + jb \Rightarrow a = \frac{1}{2}, \quad b = \frac{3}{2}$

fatorada : $V(s) = \dfrac{1}{(s-\lambda_1)(s-\lambda_2)} = \dfrac{1}{(s+\frac{1}{2}-j\frac{3}{2})(s+\frac{1}{2}+j\frac{3}{2})}$

resíduos : $c_1 = \left[(s-\lambda_1)V(s)\right]_{s=\lambda_1} = \left[\cancel{(s+\frac{1}{2}-j\frac{3}{2})} \dfrac{1}{\cancel{(s+\frac{1}{2}-j\frac{3}{2})}(s+\frac{1}{2}+j\frac{3}{2})}\right]_{s=-\frac{1}{2}+j\frac{3}{2}}$

$c_1 = \dfrac{1}{j3} = -j\dfrac{1}{3} = c_r + j c_i \Rightarrow c_r = 0, \quad c_i = -\dfrac{1}{3}$

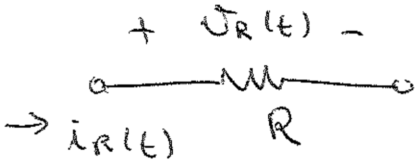
$$V(s) = \frac{\frac{2}{3} \cdot \frac{3}{2}}{(s+1/2)^2 + (3/2)^2}$$

$$v(t) = u(t)\, e^{-t/2} \operatorname{sen}\left(\frac{3}{2}t\right) \cdot \frac{2}{3}$$

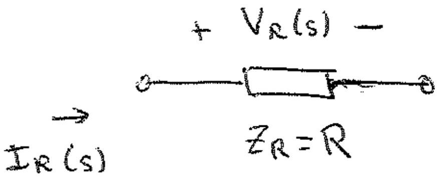
5.4 Impedância na frequência complexa.

É possível montar o sistema de equações de um circuito elétrico diretamente no domínio da frequência complexa usando as leis dos circuitos elétricos e o conceito de impedância. da mesma forma que se resolve um problema em corrente contínua.

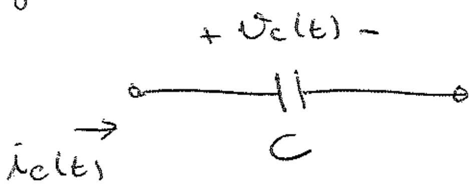
a) resistor



$$v_R(t) = R\, i_R(t) \Leftrightarrow V_R(s) = R \cdot I_R(s).$$

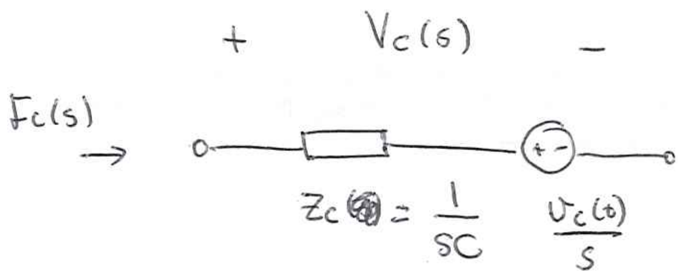


b) capacitor



$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t)\,dt \Leftrightarrow V_C(s) = \frac{1}{C}\left[\frac{I_C(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} i_C(t)\,dt\right]$$

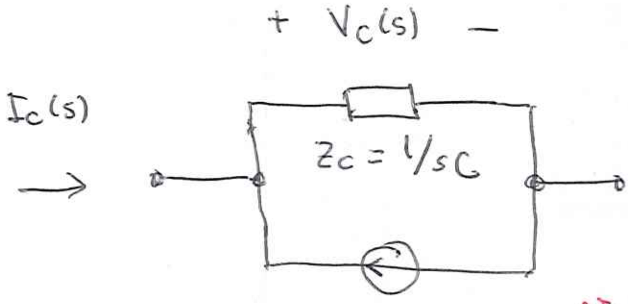
$$v_C(0) = \frac{1}{C}\int_{-\infty}^{0} i_C(t)\,dt \Rightarrow V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0)}{s}$$



$$i_c(t) = C\,\frac{dv_c(t)}{dt} \Leftrightarrow I_c(s) = C\left[s\,V_c(s) - v_c(0)\right]$$

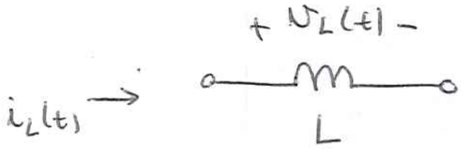
$$I_c(s) = sC\,V_c(s) - C\,v_c(0)$$

Equivalente de Norton



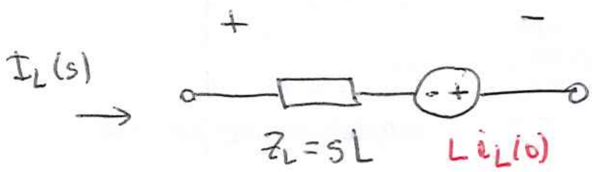
$$C\,v_c(0) = \frac{v_c(0)}{s} \cdot \frac{1}{Z_c}$$

c) indutor



$$v_L(t) = L\,\frac{d}{dt}\,i_L(t) \Leftrightarrow V_L(s) = L\left[s\,I_L(s) - i_L(0)\right]$$

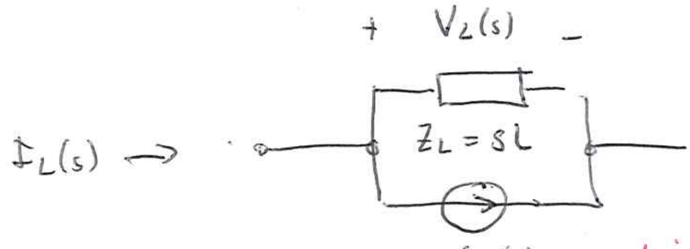
$$V_L(s) = sL \cdot I_L(s) - L\,i_L(0)$$



$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)\,dt \Leftrightarrow I_L(s) = \frac{1}{L}\left[\frac{V_L(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} v_L(t)\,dt\right]$$

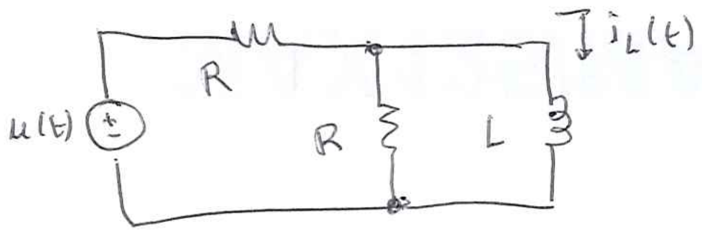
$$i_L(0) = \frac{1}{L}\int_{-\infty}^{0} v_L(t)\,dt \Rightarrow I_L(s) = \frac{V_L(s)}{sL} + \frac{i_L(0)}{s}$$

equivalente de Norton

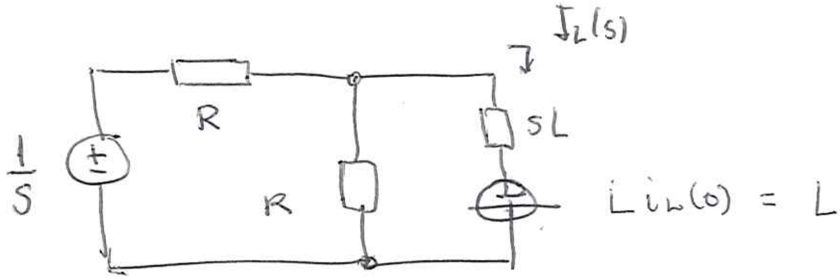


$$\frac{i_L(0)}{s} = \frac{L\,i_L(0)}{sL}$$

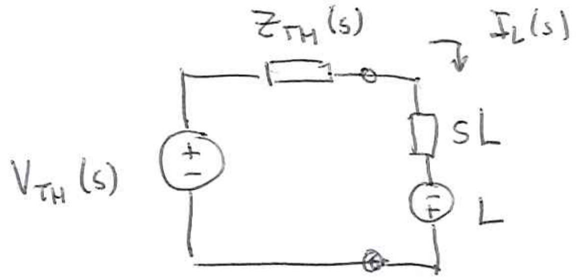
Exercício 5.4 Obtenha a expressão para a corrente elétrica $i_L(t)$ indicada no circuito elétrico abaixo considerando que $i_L(0) = 1$.



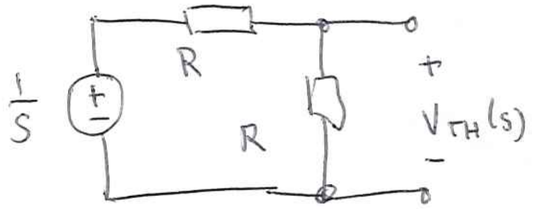
domínio s



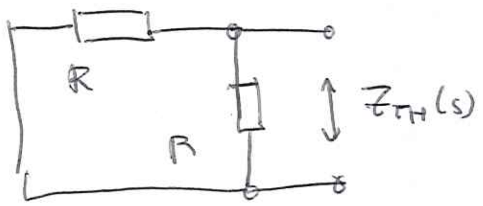
Thevenin



$$I_L(s) = \frac{V_{TH}(s) + L}{Z_{TH}(s) + sL}$$



$$V_{TH}(s) = \frac{R}{R+R}\,\frac{1}{s} = \frac{1}{2s}$$



$$Z_{TH}(s) = \frac{RR}{R+R} = \frac{R}{2}$$

$$I_L(s) = \frac{\frac{1}{2s} + L}{\frac{R}{2} + sL} \times \frac{s}{s} = \frac{sL + 1/2}{s^2 L + sR/2} \times \frac{1/L}{1/L} = \frac{s + 1/2L}{s^2 + R/2L\, s}$$

polos

$$s^2 + R/2L\, s = 0 \Rightarrow s = \begin{cases} 0 = \lambda_1 \\ -R/2L = \lambda_2 \end{cases}$$

fatorada

$$I_L(s) = \frac{s + 1/2L}{s\,(s + R/2L)}$$

resíduos

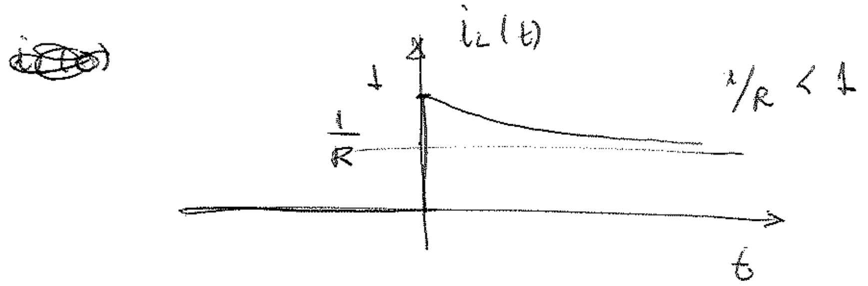
$$c_1 = \left[\cancel{s}\, \frac{s + 1/2L}{\cancel{s}\,(s + R/2L)} \right]_{s=0} = \frac{1/2L}{R/2L} = \frac{1}{R}$$

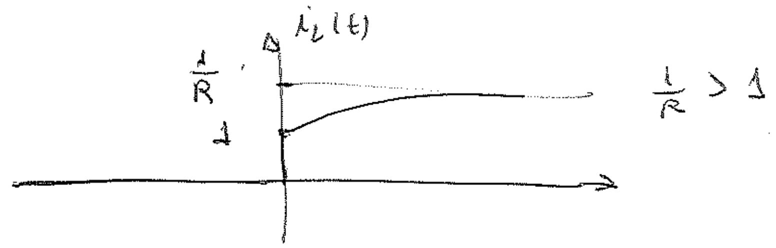
$$c_2 = \left[\cancel{(s + R/2L)}\, \frac{s + 1/2L}{s\,\cancel{(s + R/2L)}} \right]_{s=-R/2L} = \frac{-R/2L + 1/2L}{-R/2L} = 1 - \frac{1}{R}$$

transformada inversa

$$i_L(t) = u(t)\left[c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right]$$

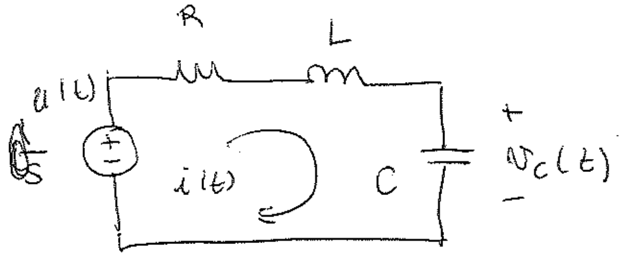
$$i_L(t) = u(t)\left[\frac{1}{R} + \left(1-\frac{1}{R}\right)e^{-tR/2L}\right]$$

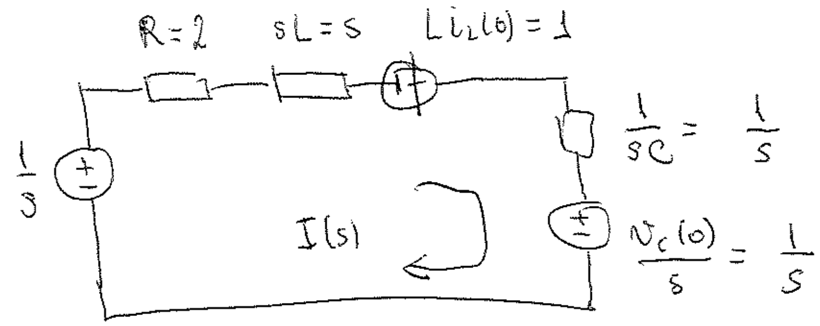




Exercício 5.5. Obtenha a expressão para a corrente elétrica $i(t)$ indicada no circuito elétrico abaixo considerando:

$i(0) = 1$, $v_c(0) = 1$, $L = 1$, $C = 1$, $R = 2$





$$I(s) = \frac{\frac{1}{s} + 1 - \frac{1}{s}}{2 + s + \frac{1}{s}} \times \frac{s}{s} = \frac{s}{s^2 + 2s + 1}$$

$$s^2 + 2s + 1 = 0 \Rightarrow s = \frac{-2 \pm \sqrt{4-4}}{2} = -1 = \lambda_1 = \lambda_2$$

$$I(s) = \frac{s}{(s+1)^2}$$

$$c_1 = \left[(s+1)^2 \frac{s}{(s+1)^2} \right]_{s=-1} = -1$$

$$c_2 = \left[\frac{d}{ds} s \right]_{s=-1} = 1$$

$$i(t) = u(t) \left[c_1 t \cdot e^{\lambda_1 t} + c_2 e^{\lambda_1 t} \right]$$

$$i(t) = u(t) \left[-t e^{-t} + e^{-t} \right]$$